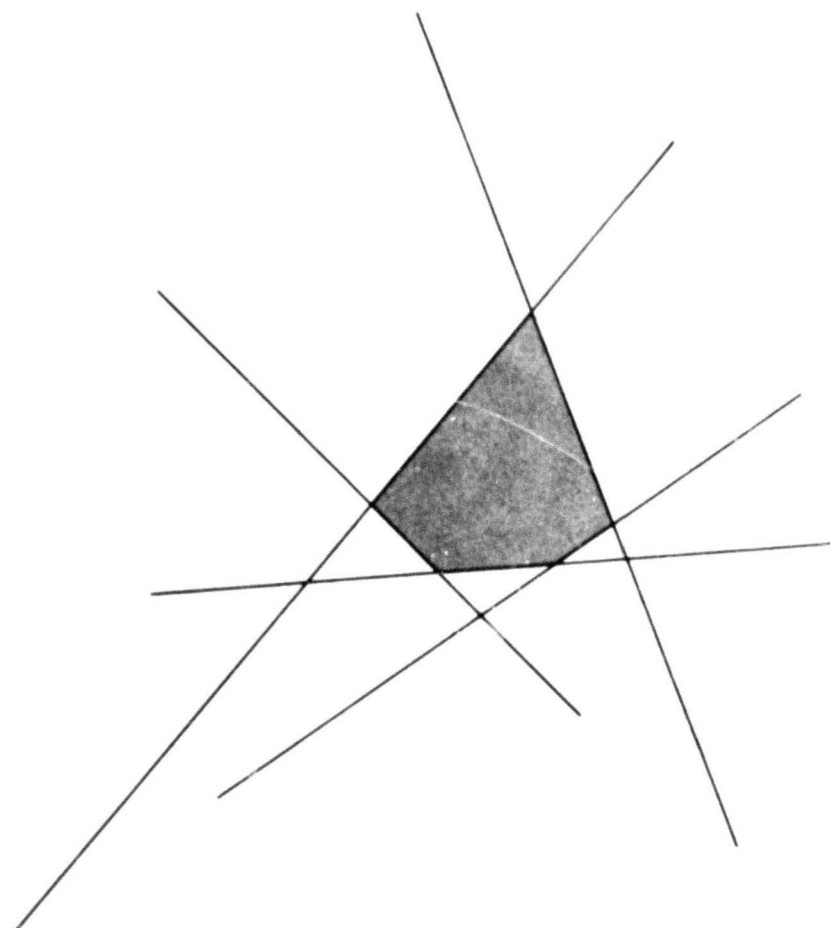


OPTIMUM ALLOCATION OF RESOURCES IN THE PURCHASE OF SPARE PARTS AND/OR ADDITIONAL SERVICE CHANNELS

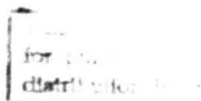
by

LEONARD J. JACOBSON

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OPTIMUM ALLOCATION OF RESOURCES IN THE PURCHASE
OF SPARE PARTS AND/OR ADDITIONAL SERVICE CHANNELS

by

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MAY 1969

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ABSTRACT

The manager of a warehouse and repair facility must decide how many spares of each of many different part types to store in inventory so as to insure, in some sense, the efficiency of the facility's operations. Customers will bring to the repair facility failed parts. If a spare of that part type is available from the warehouse, the customer will receive one and leave. Otherwise he must wait until the repair facility produces a replacement. The failed part which was brought to the repair facility will be repaired in order to replenish the inventory in the warehouse.

Having a budget constraint, and using information on the failure rates of the parts, the relative importance of each part to the other parts, and the amount of time necessary to repair a failed part, the manager must decide how to optimally allocate his resources in purchasing these spare parts. Furthermore, he might also consider allocating some of his funds in an effort to increase the capabilities of his repair facility.

The paper first formulates various mathematical models and then, in each case, shows how one can obtain a sequence of undominated solutions. () ←

INTRODUCTION AND SUMMARY

Let us suppose that there are many parts of different types in use and let us assume that each of these part types is subject to stochastic failure. We wish to set up a service facility which will repair failed parts and will also serve as a warehouse to store an inventory of spare parts. Let us say there are r different part types and that we have an initial inventory of k_i spares of type i . Assuming for the moment that $k_i > 0$, the following will occur. Whenever the first failure of a part of type i occurs, a spare from the inventory will be issued to the customer bringing it in, and the failed part will go to the repair facility. Some time later, another customer might come with a failed part of type i . He, too, will give his failed part to the repair facility. If there is a spare part of type i in the inventory, he will immediately receive it. If not, he will have to wait and he will receive the first part of type i to come out of the repair facility. If a part of type i comes out of the repair facility and no customer is queued up waiting for a part of this type, then the part goes into the inventory of type i .

Regardless of the situation with respect to other part types, if there are shortages in the inventory of part type i , each customer who is waiting for part type i --called "customers of type i "--will receive his replacement part on a first-in-first-out (FIFO) basis. However, if a customer of type j is also waiting for a part, he may or may not be given his part before those of type i receive their parts. In other words, there is no queueing discipline between types; only within types.

As an example of a "real-life" system that this paper relates to, consider a company which maintains a large fleet of aircraft. These aircraft have many different parts that are subject to wear and even failure. Such parts

might be propellers, engines, navigational equipment, flight instruments, etc. When a part fails, the customer brings the failed part to the repair facility and it is hoped that there will be a spare part of the same type to give him so as to minimize the down-time of the aircraft.

The manager of the aircraft maintenance facility must determine somehow the number of spare parts of each type to store in his warehouse.

The problem we wish to investigate then is as follows:

How can one best allocate his resources in the purchase of spare parts and/or additional service channels so as to minimize the expected number of shortages in the inventory at some future time, long after the system has been in operation?

Chapter 1 directs its attention to the case in which only spare parts and no additional service channels may be purchased. The method of approach in this chapter is as follows:

Section 1.1:

Under the assumptions given in Section 1.1.1, an expression is found for the probability that there are n parts of type i in the system. By "in the system," we mean in the repair facility, whether actually being worked on or currently queued up waiting for servicing.

We then derive an expression for the expected number of shortages of type i given that there were k_i parts of type i initially in the inventory. We will denote this by $L_i(k_i)$.

Then we show (under more general assumptions than those of Section 1.1.1) that $L_i(k_i)$ is convex in k_i .

This is followed by a discussion of the notion of dominance and the mathematical optimization problem which is formulated.

A presentation of the algorithms of Proschan and Kettelle is given.

Finally, a generalization is made to the case in which for any part type, there is only a subset of the service channels which can work on it. The possible assignment of parts to service channels is defined for a specific case and the methods of Proschan and Kettelle are shown to apply.

Section 1.2:

An analysis is presented for the case in which there is a lag time in the delivery of a failed part to the repair facility.

Section 1.3:

In this section we discuss numerical solutions to the problem as previously defined, but for systems with general arrivals to the repair facility.

Chapter 2 handles the case in which funds may be allocated to purchase both spare parts and additional service channels.

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CHAPTER 1

OPTIMUM ALLOCATION OF RESOURCES IN THE PURCHASE OF SPARE PARTS

1.1 M/M/c Case

1.1.1 Assumptions

- (1) The arrival of failed parts of type i form a Poisson process at rate λ_i . There is no lag between the time the part fails and the time it is brought into the repair facility by the customer. Therefore, the arrival of all part types to the repair facility forms a Poisson process at rate $\lambda = \sum_{i=1}^K \lambda_i$.
- (2) The repair time for all parts of all types is distributed exponentially at rate μ , and is independent of the repair time of any other part.
- (3) There are c service channels, each of which can work on parts of all types.
- (4) There is no limit imposed on the number of failed parts that can queue up in the repair facility.

1.1.2 Derivation of the Steady State Probabilities

Let $\lambda = \sum_{i=1}^K \lambda_i$; c = number of service channels; and $\rho = \lambda/c\mu$.

Assume $\rho < 1$. So far as the repair facility is concerned, there is no distinction among parts. Thus, we have an $M/M/c^{\dagger}$ queueing system in which

[†] $M/M/c$ --Standard notation in queueing theory. The first "M" denotes exponential interarrival times to the system. The second "M" denotes exponential service in the system. The "c" means there are c service channels in the repair facility. When the first "M" is replaced by "GI," it denotes a general distribution for the interarrival times, and when the second "M" is replaced by a "G," it denotes a general distribution for the service time. A "D" in place of the second "M" denotes deterministic service time.

the steady state probability distribution is given by

$$P_n = \begin{cases} P_0 \frac{(c\rho)^n}{n!} & \text{for } n \leq c \\ P_0 \frac{c^c}{c!} \rho^n & \text{for } n \geq c \end{cases}$$

where $P_0^{-1} = \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!(1-\rho)}$. Let

N_1 = number of type 1 in the system

N = number of all types together in the system, i.e., $\sum_{i=1}^r N_i = N$

$$P(N_1 = n \mid N = m) = \begin{cases} \binom{m}{n} \left(\frac{\lambda_1}{\lambda}\right)^n \left(1 - \frac{\lambda_1}{\lambda}\right)^{m-n} & \text{for } n \leq m \\ 0 & \text{for } n > m \end{cases}$$

$$\therefore P(N_1 = n) = \sum_{m=n}^{\infty} \binom{m}{n} \left(\frac{\lambda_1}{\lambda}\right)^n \left(1 - \frac{\lambda_1}{\lambda}\right)^{m-n} P_m.$$

Assume first that $n \leq c$.

$$\begin{aligned} P(N_1 = n) &= \sum_{m=n}^c \binom{m}{n} \left(\frac{\lambda_1}{\lambda}\right)^n \left(1 - \frac{\lambda_1}{\lambda}\right)^{m-n} P_0 \frac{(c\rho)^m}{m!} \\ &\quad + \sum_{m=c+1}^{\infty} \binom{m}{n} \left(\frac{\lambda_1}{\lambda}\right)^n \left(1 - \frac{\lambda_1}{\lambda}\right)^{m-n} P_0 \frac{c^c}{c!} \rho^m \end{aligned}$$

$$\begin{aligned}
&= P_o \left(\sum_{m=n}^c \frac{m!}{n!(m-n)!} \left(\frac{\lambda_i}{\lambda} \right)^n \left(1 - \frac{\lambda_i}{\lambda} \right)^{m-n} \frac{(c\rho)^n (c\rho)^{m-n}}{m!} \right. \\
&\quad \left. + \frac{c}{c!} \sum_{m=c+1}^{\infty} \frac{m!}{n!(m-n)!} \left(\frac{\lambda_i}{\lambda} \right)^n \left(1 - \frac{\lambda_i}{\lambda} \right)^{m-n} \rho^n \rho^{m-n} \right) \\
&= P_o \left(\left(\frac{\lambda_i}{\mu} \right)^n \sum_{m=n}^c \frac{\left(\frac{\lambda - \lambda_i}{\mu} \right)^{m-n}}{(m-n)!} + \frac{c}{c!} \left(\frac{\lambda_i}{\mu} \right)^n \sum_{m=c+1}^{\infty} m! \frac{\left(\frac{\lambda - \lambda_i}{c\mu} \right)^{m-n}}{(m-n)!} \right).
\end{aligned}$$

Let $\sigma_i = \frac{\lambda - \lambda_i}{c\mu}$, ε measure of the traffic intensity of all types other than type i , and let $\rho_i = \frac{\lambda_i}{\mu}$

$$P(N_i = n) = P_o \frac{\rho_i^n}{n!} \left(\sum_{u=0}^{c-n} \frac{(c\sigma_i)^u}{u!} + \frac{c}{c!} \sum_{m=c+1}^{\infty} m! \frac{\sigma_i^{m-n}}{(m-n)!} \right).$$

Now,

$$\sum_{m=c+1}^{\infty} m! \frac{\sigma_i^{m-n}}{(m-n)!} = \sum_{m=n}^{\infty} m! \frac{\sigma_i^{m-n}}{(m-n)!} - \sum_{m=n}^c m! \frac{\sigma_i^{m-n}}{(m-n)!}.$$

But,

$$\begin{aligned}
\sum_{m=n}^{\infty} m! \frac{\sigma_i^{m-n}}{(m-n)!} &= \sum_{m=n}^{\infty} m(m-1) \dots (m-n+1) \sigma_i^{m-n} = \frac{d^n}{d\sigma_i^n} \left(\sum_{m=n}^{\infty} \sigma_i^m \right) \\
&= \frac{d^n}{d\sigma_i^n} \left(\frac{\sigma_i^n}{1 - \sigma_i} \right).
\end{aligned}$$

Lemma 1.1:

$$\frac{d^n}{d\sigma^n} \left(\frac{\sigma^n}{1 - \sigma} \right) = \frac{n!}{(1 - \sigma)^{n+1}}.$$

Proof:

By induction on n . Let $n = 1$

$$\frac{d}{d\sigma} \left(\frac{\sigma}{1-\sigma} \right) = \frac{(1-\sigma)(1) - \sigma(-1)}{(1-\sigma)^2} = \frac{1}{(1-\sigma)^2}.$$

Now assume true for n and prove for $n+1$; i.e., assume

$$\begin{aligned} \frac{d^n}{d\sigma^n} \left(\frac{\sigma^n}{1-\sigma} \right) &= \frac{n!}{(1-\sigma)^{n+1}} \\ \frac{d^{n+1}}{d\sigma^{n+1}} \left(\frac{\sigma^{n+1}}{1-\sigma} \right) &= \frac{d}{d\sigma} \left[\frac{d^n}{d\sigma^n} \left(\frac{\sigma^{n+1}}{1-\sigma} \right) \right]. \end{aligned}$$

Using the general formula

$$\frac{d^n}{dx^n} [f(x)g(x)] = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

where $f(x) = x$ and $g(x) = \frac{x^n}{1-x}$ we see that

$$\frac{d}{d\sigma} \left[\frac{d^n}{d\sigma^n} \left(\frac{\sigma^{n+1}}{1-\sigma} \right) \right] = \frac{d}{d\sigma} \left[\sigma \frac{d^n}{d\sigma^n} \left(\frac{\sigma^n}{1-\sigma} \right) + n \frac{d^{n-1}}{d\sigma^{n-1}} \left(\frac{\sigma^n}{1-\sigma} \right) \right].$$

By the induction hypothesis, and using the fact that

$$\int \frac{d^n}{dx^n} [f(x)] = \frac{d^{n-1}}{dx^{n-1}} [f(x)] + K, \text{ we have}$$

$$\frac{d}{d\sigma} \left[\frac{\sigma n!}{(1-\sigma)^{n+1}} + n \int_0^\sigma \frac{n!}{(1-\sigma)^{n+1}} d\sigma \right] = \frac{d}{d\sigma} \left[\frac{\sigma n!}{(1-\sigma)^{n+1}} + \frac{n!}{(1-\sigma)^n} - n! \right]$$

$$= n! \frac{d}{d\sigma} \left[\frac{\sigma}{(1-\sigma)^{n+1}} + \frac{1-\sigma}{(1-\sigma)^{n+1}} - 1 \right] = n! \frac{d}{d\sigma} \left[\frac{1}{(1-\sigma)^{n+1}} \right] = \frac{(n+1)!}{(1-\sigma)^{n+2}}$$

$$\therefore \frac{d^{n+1}}{d\sigma^{n+1}} \left(\frac{\sigma^{n+1}}{1-\sigma} \right) = \frac{(n+1)!}{(1-\sigma)^{n+2}} \text{ and the lemma is proved.}$$

Therefore, for $n \leq c$,

$$\begin{aligned} P(N_1 = n) &= P_0 \frac{\rho_1^n}{n!} \left[\sum_{u=0}^{c-n} \frac{(c\sigma_1)^u}{u!} + \frac{c^{c-n}}{c!} \left(\frac{n!}{(1-\sigma_1)^{n+1}} - \sum_{m=n}^c m! \frac{\sigma_1^{m-n}}{(m-n)!} \right) \right] \\ &= P_0 \frac{\rho_1^n}{n!} \left[\sum_{u=0}^{c-n} \frac{(c\sigma_1)^u}{u!} + \frac{c^{c-n}}{c!} \left(\frac{n!}{(1-\sigma_1)^{n+1}} - \sum_{u=0}^{c-n} (u+n)! \frac{\sigma_1^u}{u!} \right) \right] \\ &= P_0 \frac{\rho_1^n}{n!} \left[\sum_{u=0}^{c-n} \left(\frac{(c\sigma_1)^u}{u!} - \frac{c^{c-n}}{c!} (u+n)! \frac{\sigma_1^u}{u!} \right) + \frac{c^{c-n} n!}{c! (1-\sigma_1)^{n+1}} \right]. \end{aligned}$$

Now if $n > c$, the reader can see that

$$P(N_1 = n) = P_0 \frac{\rho_1^n}{n!} \frac{c^{c-n}}{c!} \sum_{m=n}^{\infty} m! \frac{\sigma_1^{m-n}}{(m-n)!} = P_0 \frac{\rho_1^n}{n!} \frac{c^{c-n}}{c!} \frac{1}{(1-\sigma_1)^{n+1}}.$$

Thus, we have proved the following theorem:

Theorem 1.1:

The steady state probabilities of the system under the assumptions of Section 1.1.1 are given by

$$(1) \quad P(N_1 = n) = \begin{cases} P_0 \frac{\rho_1^n}{n!} \left[\sum_{u=0}^{c-n} \left(\frac{(c\sigma_1)^u}{u!} - \frac{c^{c-n}}{c!} (u+n)! \frac{\sigma_1^u}{u!} \right) + \frac{c^{c-n} n!}{c! (1 - \sigma_1)^{n+1}} \right] & \text{for } n \leq c \\ P_0 \frac{\rho_1^n}{c!} \frac{c^{c-n}}{(1 - \sigma_1)^{n+1}} & \text{for } n \geq c \end{cases}$$

1.1.3 The Expected Number of Shortages

Let

$$L_1(k_1) = \sum_{n=1}^{\infty} n P(N_1 = n + k_1).$$

Assume $k_1 \leq c - 1$

$$\begin{aligned} L_1(k_1) &= \sum_{n=1}^{c-k_1} n \left[P_0 \frac{\rho_1^{n+k_1}}{(n+k_1)!} \left[\sum_{u=0}^{c-n-k_1} \left(\frac{(c\sigma_1)^u}{u!} - \frac{c^{c-n-k_1}}{c!} (u+n+k_1)! \frac{\sigma_1^u}{u!} \right) \right. \right. \\ &\quad \left. \left. + \frac{c^{c-n-k_1} (n+k_1)!}{c! (1 - \sigma_1)^{n+k_1+1}} \right] \right] + \sum_{n=c-k_1+1}^{\infty} n \left[P_0 \frac{\rho_1^{n+k_1}}{c!} \frac{c^{c-n-k_1}}{(1 - \sigma_1)^{n+k_1+1}} \right] \\ &= \sum_{n=1}^{c-k_1} n P_0 \frac{\rho_1^{n+k_1}}{(n+k_1)!} \sum_{u=0}^{c-n-k_1} \left(\frac{(c\sigma_1)^u}{u!} - \frac{c^{c-n-k_1}}{c!} (u+n+k_1)! \frac{\sigma_1^u}{u!} \right) \\ &\quad + \sum_{n=1}^{\infty} n P_0 \frac{\rho_1^{n+k_1}}{c!} \frac{c^{c-n-k_1}}{(1 - \sigma_1)^{n+k_1+1}}. \end{aligned}$$

Now the second term is equal to

$$P_0 \frac{c^c}{c!} \frac{1}{1 - \sigma_1} \sum_{n=1}^{\infty} n \left[\frac{\rho_1}{c(1 - \sigma_1)} \right]^{n+k_1} = \frac{P_0 c^c \theta_1^{k_1+1}}{c!(1 - \sigma_1)} \sum_{n=1}^{\infty} n \theta_1^{n-1}$$

where $\theta_1 = \frac{\rho_1}{c(1 - \sigma_1)}$. But

$$\sum_{n=1}^{\infty} n \theta_1^{n-1} = \frac{d}{d\theta_1} \left(\sum_{n=1}^{\infty} \theta_1^n \right) = \frac{d}{d\theta_1} \left(\frac{\theta_1}{1 - \theta_1} \right) = \frac{1}{(1 - \theta_1)^2}.$$

Thus, the second term equals $\frac{P_0 c^c \theta_1^{k_1+1}}{c!(1 - \theta_1)^2(1 - \sigma_1)}$. Now if

$$k_1 \geq c, L_1(k_1) = \sum_{n=1}^{\infty} n \left[P_0 \frac{\rho_1^{n+k_1}}{c!} \frac{c^{c-n-k_1}}{(1 - \sigma_1)^{n+k_1+1}} \right].$$

But this is simply the second term that was just computed. We can now write out a general formula for $L_1(k_1)$.

$$(2) \quad L_1(k_1) = \begin{cases} P_0 \left(\sum_{n=1}^{c-k_1} n \frac{\rho_1^{n+k_1}}{(n+k_1)!} \sum_{u=0}^{c-n-k_1} \left(\frac{(c\sigma_1)^u}{u!} - \frac{c^{c-n-k_1}}{c!} (u+n+k_1)! \frac{\sigma_1^u}{u!} \right) + \frac{c^c \theta_1^{k_1+1}}{c!(1 - \theta_1)^2(1 - \sigma_1)} \right) & \text{for } k_1 \leq c-1 \\ P_0 \frac{c^c \theta_1^{k_1+1}}{c!(1 - \theta_1)^2(1 - \sigma_1)} & \text{for } k_1 \geq c-1 \end{cases}$$

1.1.4 Proof of the Convexity of $L(k)$ in k

Theorem 1.2:

Under completely general assumptions on the distribution of the interarrival times of parts to the repair facility and on the distribution of the service times, and allowing any number of service channels from 1 to ∞ , the expected number of shortages of a part type is convex in the number of spares initially in the inventory.

Proof:

Let

k = # of spares initially in the inventory

P_n = $P\{n \text{ parts in the system (steady or transient state)}\}$

$$L(k) = \sum_{n=1}^{\infty} n P_{n+k}$$

$$\Delta L(k) = L(k+1) - L(k)$$

$$\Delta^2 L(k) = L(k+2) - 2L(k+1) + L(k)$$

$$= \sum_{n=1}^{\infty} n [P_{n+k+2} - 2P_{n+k+1} + P_{n+k}]$$

$$= \begin{array}{l} -P_{k+3} - 2P_{k+2} + P_{k+1} \\ + 2P_{k+4} - 4P_{k+3} + 2P_{k+2} \\ + 3P_{k+5} - 6P_{k+4} + 3P_{k+3} \\ \vdots \\ + (n-1)P_{k+n+1} - 2(n-1)P_{k+n} + (n-1)P_{k+n-1} \\ + nP_{k+n+2} - 2nP_{k+n+1} + nP_{k+n} \\ + (n+1)P_{k+n+3} - 2(n+1)P_{k+n+2} + (n+1)P_{k+n+1} \\ \vdots \end{array}$$

Note that along the indicated diagonals, the sum is zero. Thus, the total sum is merely $P_{k+1} \geq 0$. //

1.1.5 The Optimization Problem

1.1.5.1 Notion of Dominance

For each allocation $\underline{k} = (k_1, k_2, \dots, k_r)$, there is a value (in the sense of "worth") and a cost associated with it, denoted by $V(\underline{k})$ and $C(\underline{k})$ respectively.

Definition: Allocation \underline{k}_1 is said to *dominate* allocation \underline{k}_2 iff

$$V(\underline{k}_1) \geq V(\underline{k}_2) \text{ and } C(\underline{k}_1) \leq C(\underline{k}_2) .$$

Definition: Allocation \underline{k}_1 is said to *strictly dominate* allocation \underline{k}_2 iff at least one of the above inequalities is strict.

Definition: Allocation \underline{k}_1 is said to be *undominated* if there does not exist an allocation \underline{k}_2 which strictly dominates it.

Definition: A sequence S of undominated allocations is called a *complete family of undominated allocations* iff for every allocation \underline{k} , either $\underline{k} \in S$ or there exists an element of S which dominates \underline{k} .

Definition: A sequence S of undominated allocation, which is not a complete family, is called an *incomplete family of undominated allocations*.

1.1.5.2 Mathematical Formulation of the Optimization Problem

Let us say that each part of type i has a cost c_i . Thus, allocation \underline{k} has a cost associated with it of $C(\underline{k}) = \sum_{i=1}^r c_i k_i$. Furthermore, let us

say that there is a constant v_i associated with parts of type i which is a measure of the relative importance of part type i which is a measure of the relative importance of part type i to the other parts. For example, v_i might be the cost in dollars for a shortage of part type i .

We wish to construct a sequence of undominated allocations to the problem with $V(\underline{k}) = -W(\underline{k})$, where

$$(3) \quad W(\underline{k}) = \sum_{i=1}^r v_i L_i(k_i)$$

and

$$(4) \quad C(\underline{k}) = \sum_{i=1}^r c_i k_i.$$

Having this sequence of undominated allocations, one knows that if \underline{k}^* is an element of the sequence then for any allocation \underline{k}' with $W(\underline{k}') < W(\underline{k}^*)$, then it must be the case that $C(\underline{k}') \geq C(\underline{k}^*)$.

If the sequence is an incomplete family of undominated allocations, the following problem arises. If exactly b dollars are allotted to be spent purchasing spare parts, and if in constructing the incomplete family of undominated allocations one finds no solution with associated cost exactly equal to b , but finds the two solutions \underline{k}_1 and \underline{k}_2 such that $C(\underline{k}_1) < b < C(\underline{k}_2)$, and $C(\underline{k}_1)$ is closest to b from below while $C(\underline{k}_2)$ is closest to b from above, then the optimal allocation vector to the problem

$$(5) \quad \begin{array}{ll} \text{Minimize} & W(\underline{k}) \\ \text{Subject to} & C(\underline{k}) \leq b \end{array}$$

might be \underline{k}_1 , but it is possible that there is an undominated allocation which the algorithm skipped over and this allocation has cost between $C(\underline{k}_1)$

and b . In any case, if we choose allocation \underline{k}_1 as the optimum vector of allocations to solve (5) above, then our error in the functional value is no greater than $W(\underline{k}_2) - W(\underline{k}_1)$ and the error in the cost is no greater than $b - C(\underline{k}_1)$.

If, however, the generated sequence is a complete family of undominated allocations, the problem just stated does not arise. That is, the optimal vector of allocations to (5) is \underline{k}_1 .

1.1.6 Two Algorithms to Generate Sequences of Undominated Allocations

1.1.6.1 Proschan's Algorithm

The following algorithm is taken directly from reference [1]. The notation and wording has been only slightly changed to be more meaningful to this problem.

Start with the cheapest cost allocation $\underline{k} = (0, 0, \dots, 0)$. Obtain successively more expensive allocations as follows. If our present allocation is \underline{k} , we determine the index, say i_0 , for which

$$\frac{v_i}{c_i} \Delta L_i(k_i)$$

is minimum over $i = 1, 2, \dots, r$. (If the minimum is achieved for more than one value of the index, choose the lowest among these.) Then the next allocation is $(k_1, k_2, \dots, k_{i_0-1}, k_{i_0}+1, k_{i_0+1}, \dots, k_r)$; that is, we have added a single unit of the i_0 -th type to \underline{k} .

The " Δ " sign in the above expression is defined to be

$$\Delta L_i(k_i) = L_i(k_i + 1) - L_i(k_i).$$

It can be shown that for the function $L_i(k_i)$ as given in Equation (2) (where the subscript "i" has been dropped),

$$(6) \Delta L(k) = \begin{cases} P_0 \left\{ \sum_{n=1}^{c-k-1} n \frac{\rho^{n+k}}{(n+k)!} \sum_{u=0}^{c-n-k-1} \left[\left(\frac{\rho}{n+k+1} - 1 \right) \frac{(\sigma)^u}{u!} \right. \right. \\ \quad \left. \left. - \left(\frac{\rho(u+n+k+1)}{n+k+1} - c \right) \frac{c^{c-n-k-1}}{c!} (u+n+k)! \frac{\sigma^u}{u!} \right] \right. \\ \quad \left. - \frac{c^c \theta^{k+1}}{c!(1-\theta)(1-\sigma)} \right\} \text{ for } 0 \leq k \leq c-2 \\ \\ - \frac{P_0 c^c \theta^{k+1}}{c!(1-\theta)(1-\sigma)} \text{ for } k \geq c-1. \end{cases}$$

Proschan has shown that if $W(\underline{k}) = \sum_{i=1}^r v_i L_i(k_i)$ is convex in \underline{k} , then the previously stated algorithm produces an incomplete family of undominated allocations. Since Section 1.1.4 shows that $L_i(k_i)$ is convex in k_i , we know that $W(\underline{k})$ is convex in \underline{k} . Thus, the algorithm applies.

It should be noted that Proschan's algorithm will handle multiple constraints. But since this does not add to the purpose of this paper, a discussion of multiple constraints is omitted.

1.1.6.2 Kettelle's Algorithm

In this section, we construct a complete family of undominated allocations for the problem in which $W(\underline{k})$ and $C(\underline{k})$ are given by (3) and (4), respectively. The method employed is that of Kettelle.

The algorithm works progressively in that it first obtains a complete family of undominated allocations for the subsystem in which only two of the r part types are present. Then it takes another two of the r part types and constructs a complete family of undominated allocations for this subsystem. Now it takes these two complete families and joins them in such a way as to generate from them a complete family for the subsystem

containing all four of the part types. The method proceeds in that manner so that at each stage, a complete family is generated for successively larger subsystems from two smaller subsystems. There is no requirement that the subsystems be of equal size. Nor is there any requirement of convexity of $W(k)$ as there was in Proschan's algorithm.

For the exact algorithm at each stage, the reader is referred to Kettelle [3].

It should be noted that Kettelle's algorithm is applicable only when there is one cost constraint. Proschan and Bray [7] have extended the method, however, to multiple constraints.

As in the previous section, a discussion of the multiple constraint case is omitted.

1.1.7 A Simple Extension

Let γ_i represent the i -th service channel.

Let $T = \{\text{all service channels}\} = \{\gamma_i \mid i = 1, 2, \dots, c\}$

$T(i) = \{\gamma_j \mid \text{service channel } j \text{ can service part type } i\}$

Assume the following:

- (1) $T(i) \neq \emptyset$ for all $i = 1, 2, \dots, r$.
- (2) For all j , there exists i such that $\gamma_j \in T(i)$.
- (3) $T(i) \subset T$ and $T(i) \neq T$ for all $i = 1, 2, \dots, r$.
- (4) If $T(i) \cap T(j) \neq \emptyset$, then $T(i) = T(j)$ for all $i, j = 1, 2, \dots, r$.

These assumptions result in a unique partition of T into q disjoint subsets,

$$T = T_1 + T_2 + \dots + T_q .$$

This partition then separates the entire system into q completely independent queueing systems.

Let each subset T_i have a parameter μ_i associated with it which is the rate at which all $\gamma_j \in T_i$ produce exponential service.

We now wish to find a sequence of undominated allocations to this problem with

$$(7) \quad \begin{aligned} W(\underline{k}) &= \sum_{i=1}^q \sum_{j \in T_i} v_j L_j(k_j) \\ C(\underline{k}) &= \sum_{i=1}^q \sum_{j \in T_i} c_j k_j . \end{aligned}$$

It is easy to see that the form of (7) in no way complicates the method of solution as outlined in either algorithm of Section 1.1.6.

Thus, the methods of both Proschan and Kettelle readily apply to this extension of the problem.

1.2 Lag in Delivery

Suppose that upon failure of a part type, the failed part does not appear instantaneously at the repair facility. Rather, let us assume that there is a lag time D , a random variable with general distribution $H(x)$ (assumed different for each part type).

As in Section 1.1, let us assume that the occurrence stream of failures of part type i forms a Poisson process at rate λ_i . We are interested in determining the arrival stream to the repair facility.

Dropping the subscript i for the moment, let us define

S_j = time the j -th failure (of part type i) occurs

D_j = lag time for j -th failure before it appears at the repair facility

Y_j = time of the j -th arrival to the repair facility (not necessarily the same as the time the j -th failure arrives at the repair facility).

The following figure represents a possible outcome in the failure and arrival pattern.

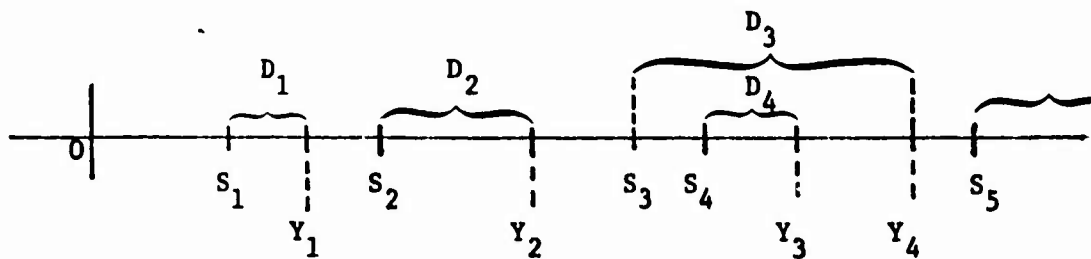


FIGURE 1.1: POSSIBLE OUTCOME IN THE FAILURE AND ARRIVAL PATTERN

We must determine the distribution of $Y_{j+1} - Y_j$, the interarrival time of failed parts to the repair facility. To do this, consider an imaginary $M/G/\infty$ queueing system. The occurrence of a part failing will be an arrival to the imaginary system. In this imaginary system, the part goes immediately into service, no matter how many parts have already arrived (i.e., failed); thus, there are an infinite number of servers in the imaginary system. The service is analogous to the lag time D . When the part leaves the imaginary system, it has completed its lag time and thus it arrives at our actual repair facility.

Therefore, we can say that the arrival stream to our repair facility is the output stream of an $M/G/\infty$ queueing system, where the service distribution is $H(x)$.

Mirasol [4] has shown that the output stream of an $M/G/\infty$ queueing system is a nonhomogeneous Poisson process such that if $\psi(t, T)$ = number

of customers to leave during $(t, t + T]$, then

$$\psi(t, T) = \text{Poisson} \left(\lambda \int_t^{t+T} G(x) dx \right).$$

Taking the limit as $t \rightarrow \infty$ of the parameter shows that in steady state, the output stream is a Poisson process at rate λ , the input rate.

Thus, even under these more general conditions, the interarrival times of failed parts to the system is exponential at rate λ . Introducing the subscript i again, we can say that the arrival stream of parts of type i to the repair facility is a Poisson process at rate λ_i .

Let us say that for part type i , $D \sim H_i(x)$ and let us let

$$\sigma_i = \int_0^{\infty} x dH_i(x), \text{ the expected value of } D \text{ for the } i\text{-th part type. Let}$$

$w_i(k_i)$ = the expected waiting time a customer incurs between the time the failed part arrives to the repair facility and the time a replacement is available for the customer, given that k_i parts of type i are initially in the inventory. Finally, let $\tau_i = \sigma_i + w_i(k_i)$, the expected total time a customer of type i must wait from the moment his part fails to the time when he has a replacement from the repair facility.

Let us assume the following:

- (1) There are an infinite number of service channels.
- (2) The service time required to repair a part of type i is a random variable with general distribution $G_i(x)$.

Under these assumptions, Proschan [6] has shown that the expected number of shortages (in steady state) of parts of type i , given k_i of them are initially in the inventory is given by

$$(8) \quad L_1(k_1) = e^{-\lambda_1 \eta_1} \sum_{j=k_1+1}^{\infty} (j - k_1) \frac{(\lambda_1 \eta_1)^j}{j!}$$

$$\text{where } \eta_1 = \int_0^{\infty} x dG_1(x) .$$

In order to find $w_1(k_1)$, observe the following:

We define an imaginary queueing system (not the one previously defined in this section) in which the steady state probabilities I_n are given by

$$I_n = \begin{cases} \sum_{j=0}^{k_1-1} P_j & n = 0 \\ P_{n+k_1} & n > 0 \end{cases}$$

where P_n is the steady state probability that there are n of type 1 in the system.

Let J = the expected number in the imaginary system. Using the well known relation " $L = \lambda w$ " (see [2]) where L represents the expected number in the system, λ the arrival rate of customers to the system, and w the expected waiting time a customer incurs in the system, we have that the expected waiting time in our imaginary system is J/λ_1 . But $J = L_1(k_1)$. Therefore,

$$(9) \quad w_1(k_1) = L_1(k_1)/\lambda_1 .$$

If τ_1 , as defined earlier in this section, is the expected total time a customer of type 1 must wait from the moment his part fails to the time when he has a replacement from the repair facility, and if the

probability that an arbitrary customer is of type i is λ_i/λ , then the expected total waiting time of an arbitrary customer is given by

$$\sum_{i=1}^r \frac{\lambda_i \tau_i}{\lambda} = \left(\sum_{i=1}^r \lambda_i \sigma_i + \sum_{i=1}^r \lambda_i w_i(k_i) \right) / \lambda .$$

Introducing the constants v_i as was done previously, and using (9),

$$(10) \quad W(\underline{k}) = \left(\sum_{i=1}^r \lambda_i v_i \sigma_i + \sum_{i=1}^r v_i L_i(k_i) \right) / \lambda .$$

Since in Section 1.1.4 $L_i(k_i)$ is convex in k_i , it follows that $W(\underline{k})$ is convex in \underline{k} . This being the case, both the algorithm of Proschan and that of Kettelle apply.

It is interesting to note that whether one applies one algorithm or the other, the resulting sequence of undominated allocations will be identical to the resulting sequence if he had applied the same algorithm to the problem of minimizing the weighted sum of the expected number of shortages. That this is true is evident from (10).

1.3 Systems with General Arrivals

1.3.1 GI/D/ ∞ Case

Assume the distribution of the interarrival times to the repair facility to be $G_i(x)$ for the i -th part type. Assume further that the service distribution for this part type is deterministic at value d_i . Finally, assume that there are an infinite number of servers available to work on failed parts. Thus, the underlying queueing system is assumed to be GI/D/ ∞ .

Let us solve for the steady state probabilities of this system.

Let $P_n(t) = P(n \text{ parts of type } i \text{ in system at time } t)$. Then,

$P_n(t + d_1) + \sum_{j=0}^{\infty} P_j(t) \cdot P\{n \text{ parts of type 1 arrive during the interval } (t, t + d_1]\}$. Since the infinite sum above is one, we have as $t \rightarrow \infty$

$P_n = P\{n \text{ parts of type 1 arrive during an interval of length } d_1, \text{ "long after" the system has been in operation}\}$.

Assume initially that $n \geq 1$. Let us condition on the arrival of the first of these n arrivals. Suppose it occurs x units of time after the interval has started, and since $n \geq 1$, $0 \leq x < d_1$. Then during the remainder of the interval, $d_1 - x$, exactly $n - 1$ arrivals must occur. Figure 1.2 illustrates this discussion.

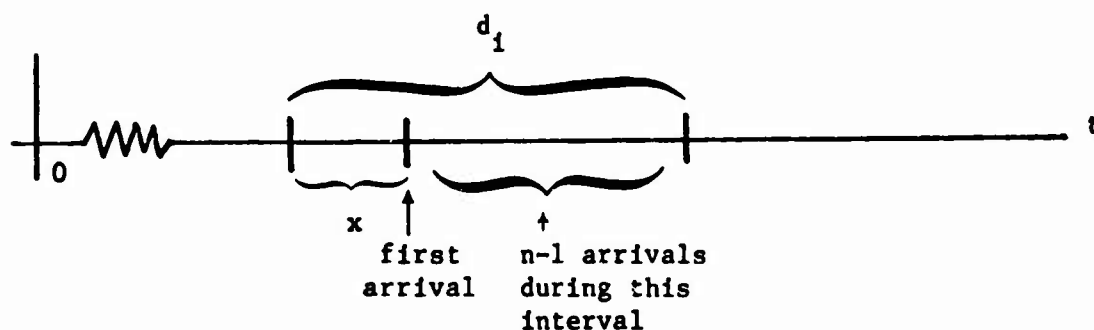


FIGURE 1.2: ARRIVALS DURING INTERVAL OF LENGTH d_1

Now the probability of exactly $n - 1$ arrivals during the interval of length $d_1 - x$ is given by

$$G_1^{(n-1)}(d_1 - x) - G_1^{(n)}(d_1 - x)$$

where $G_1^{(n)}$ is the n -fold convolution of G_1 with itself.

In order to determine the distribution of x , we note that x is the "residual life" or "excess time" until another arrival occurs. It is known

(see [8]) that in steady state, the excess time has the "equilibrium distribution" of G_1 , denoted by G_1^e , where

$$G_1^e(t) = \frac{1}{\nu_1} \int_0^t [1 - G_1(u)] du$$

and $\nu_1 = \int_0^\infty u dG_1(u)$.

Thus, unconditioning the above probability expression, we have for $n \geq 1$,

$$(11.a) \quad P_n = \int_0^{d_1} \left[G_1^{(n-1)}(d_1 - x) - G_1^{(n)}(d_1 - x) \right] dG_1^e(x).$$

For $n = 0$,

$$(11.b) \quad P_0 = 1 - G_1^e(d_1).$$

The expected number of shortages of part type 1 is given by

$$(12) \quad L_1(k_1) = \sum_{n=1}^{\infty} n \left(\int_0^{d_1} \left[G_1^{(n+k_1-1)}(d_1 - x) - G_1^{(n+k_1)}(d_1 - x) \right] dG_1^e(x) \right).$$

Since by Section 1.1.4, $L_1(k_1)$ is convex in k_1 , we know that $W(\underline{k}) = \sum_{i=1}^r v_i L_i(k_i)$ is convex in \underline{k} and either of the algorithms of section 1.1.6 apply to generate a sequence of undominated allocations.

As done in Section 1.2, we could apply " $L = \lambda W$ " if we desired to minimize the expected waiting time instead of the expected number of shortages. However, as was the case in Section 1.2, the resulting optimal allocation

vector, \underline{k}^* , would be identical to the optimal allocation vector obtained in minimizing the expected number of shortages.

It should be mentioned that even though one might have an analytical expression for $G_1(x)$, he will not, in general, be able to get a working expression for $G_1^{(n)}(x)$, or for that matter, Equations (11.a) and (11.b). Therefore, he will have to resort to a numerical solution to solve (11). This, though possibly time consuming even on a high speed computer, should not present too great a difficulty.

1.3.2 GI/G/c--Simulation Studies

In some cases, it will not be feasible to make the simplifying assumptions of the previous sections with regards to the arrival stream and/or the distribution of service times. Unless one is able to derive an analytical expression for $\{P_n\}$ under his particular set of assumptions, he will be forced to use simulation. If one uses simulation, there is, of course, no restriction on the number of service channels he may specify for the model.

Having obtained the $\{P_n\}$ through simulation, one may then numerically proceed with the analysis to obtain an expression for $W(k)$, and, by virtue of Section 1.1.4, he may apply either of the algorithms of Section 1.1.6.

CHAPTER 2

OPTIMUM ALLOCATION OF RESOURCES IN THE PURCHASE OF SPARE PARTS AND ADDITIONAL SERVICE CHANNELS

2.1 Introduction

Consider the problem of Section 1.1 with the following change. Instead of being allowed to spend the resources only on spare parts of each type, we are now allowed to allocate an unspecified proportion of the money to purchase (or create) additional service channels. How then should we allocate our resources?

2.2 Algorithm

We assume we start with c service channels and the cost of these is zero. Let s be the number of additional channels, and let $Q(s)$ be the cost to purchase them. Two assumptions on $Q(s)$ are made:

- (1) $\Delta Q(s) = Q(s + 1) - Q(s) > 0$ for all $s = 0, 1, 2, \dots$
- (2) $\lim_{s \rightarrow \infty} Q(s) = \infty$

In view of Assumption (2), and since it is not unreasonable to assume a space constraint, let us say that under no circumstances will we purchase more than q additional service channels.

We now wish to construct a complete family of undominated allocations to this generalized problem. The algorithm that we will employ is as follows:

- I. Using Kettelle's algorithm, assume $c + s$ service channels, and construct a complete family of undominated allocations. Do this for $s = 0, 1, 2, \dots, q$.
- II. Construct the following table:

# of Service Channels	Zero-th Allocation	First Allocation		n-th Allocation
c	(0,0 ,w ₀₀)	(k ₀₁ ,b ₀₁ ,w ₀₁)	(k _{0n} ,b _{0n} ,w _{0n}). . .
c + 1	(0,b ₁₀ ,w ₁₀)	(k ₁₁ ,b ₁₁ ,w ₁₁)	(k _{1n} ,b _{1n} ,w _{1n}). . .
c + 2	(0,b ₂₀ ,w ₂₀)	(k ₂₁ ,b ₂₁ ,w ₂₁)	(k _{2n} ,b _{2n} ,w _{2n}). . .
.	.	.		.
.	.	.		.
.	.	.		.
c + q	(0,b _{q0} ,w _{q0})	(k _{q1} ,b _{q1} ,w _{q1})	(k _{qn} ,b _{qn} ,w _{qn}). . .

where \underline{k}_{in} is the n-th member of the complete family of undominated allocations, assuming c + i service channels, b_{in} is the cost of the allocation vector \underline{k}_{in} plus the cost Q(i) of i additional service channels, and w_{in} is the functional value $W(\underline{k}_{in})$.

III. t = 1

$$(\underline{k}_t, b_t, w_t) = (0, 0, w_{00})$$

$$j_0 = 1 ; j_i = 0 \text{ for all } i = 1, 2, \dots, q .$$

IV. i = 0. Compare w_t to w_{ij_i}

(a) If $w_t \leq w_{ij_i}$, set $j_i = j_i + 1$ and repeat IV-(a)

(b) If $w_t > w_{ij_i}$, continue

$$i = i + 1$$

If i > q, go to V

Otherwise, go to IV-(a) .

V. $t = t + 1$

Let s be the index such that

$$b_{sj_s} = \min_i \{b_{ij_i}\}.$$

If there is a tie, choose s such that w_{sj_s} is the minimum of all those with equal cost.

$$\text{Set } (k_t, b_t, w_t) = (k_{sj_s}, b_{sj_s}, w_{sj_s})$$

$$j_s = j_s + 1$$

Go to Step IV .

The resulting sequence represents a complete family of undominated allocations for the generalized problem, as proved in Theorem 2.1.

Theorem 2.1:

The allocations obtained using I-V constitute a complete family of undominated allocations.

Proof:

First we will show that all allocations found by the procedure are undominated.

Assume the contrary. Say there exists m, t such that $w_m < w_t$ and $b_m < b_t$, and both of these solutions were picked by the procedure.

Steps IV and V tell us this is not possible because if allocation t were chosen, then any succeeding choice would have higher cost and lower functional value.

Now suppose m is such that allocation m was not chosen by the procedure. And let us say that allocation m dominates allocation z which the procedure chose. Denote this by

$$A_m > A_z .$$

By the definition of dominance in Section 1.1.6.1,

$$A_m > A_z \Leftrightarrow W_m < W_z \text{ and } b_m < b_z .$$

Let the sequence of chosen allocations be

$$S = \{A_1, A_2, \dots, A_n, A_{n+1}, \dots, A_z, \dots\} .$$

Let n be such that $b_n < b_m < b_{n+1}$.

Since $W_z < W_{n+1}$ and $W_m < W_z$, we have $W_m < W_{n+1}$ which implies A_m dominates all allocations from A_{n+1} to A_z .

When A_{n+1} was chosen as the allocation after A_n , either A_m was "checked" or it was not. By "checked," we mean that in Step V, A_m was one of the $q + 1$ allocations which were compared. If A_m was indeed checked, then by virtue of this fact,

$$(13) \quad W_m < W_n \text{ and } b_m > b_n .$$

Since A_{n+1} was also checked,

$$(14) \quad W_{n+1} < W_n \text{ and } b_{n+1} > b_n$$

Since by Assumption $A_m > A_{n+1}$,

$$(15) \quad W_m < W_{n+1} < W_n \text{ and } b_n < b_m < b_{n+1} .$$

But if this is true, A_m would have been chosen over A_{n+1} , and A_{n+1} would never have been chosen. Thus, A_m could not have been checked.

Now if A_m was not checked, there are two possibilities.

- (i) A_m was already discarded as a possible choice.
- (ii) A_m has not yet been checked.

Suppose (i) is the case. But since $W_m < W_n$, A_m could not have been discarded because Step IV of the procedure will discard A_m only if $W_n \leq W_m$. Therefore, (i) is not possible.

Suppose (ii) occurs. This means there exists A_v in the same row of the table as A_m such that

$$(16) \quad W_m < W_v \text{ and } b_m > b_v.$$

Let us say that A_v is being checked as a possibility to become A_{n+1} .

First note that A_v could not have been the one chosen for A_{n+1} because $A_m > A_{n+1}$, but $A_m \not\leq A_v$.

Now since A_v was checked, the algorithm says

$$(17) \quad W_v < W_n \text{ and } b_v > b_n.$$

And since A_{n+1} was checked and chosen over A_v ,

$$(18) \quad W_{n+1} < W_n \text{ and } b_{n+1} > b_n$$

and

$$(19) \quad b_{n+1} < b_v$$

Inequalities (16) - (19) imply

$$b_m > b_v > b_{n+1} > b_n.$$

But we have already said that $b_{n+1} > b_m > b_n$.

Thus, (ii) is not possible and we have shown that no allocation in the table dominates a chosen allocation.

In order to prove that all chosen allocations are undominated, it remains to show that if we have an allocation A_m which Kettelle's algorithm did not choose when Step I was performed, then A_m could not dominate any chosen allocation in S .

Assume otherwise. That is, there exists $A_z \in S$ such that $A_m > A_z$. Since Kettelle's algorithm did not choose A_m , there exists A_t , an allocation with the same number of additional service channels as in allocation A_m , such that $A_t > A_m$.

Now $A_t \notin S$, for if it were, then since

$$A_t > A_m > A_z,$$

we would have two chosen allocations in S , one of which dominates the other. This is not possible.

But even if A_t were not chosen, we have just proved that no allocation in the table (such as A_t) dominates a chosen allocation in S (such as A_z).

Now we must show the converse; i.e., if $A_m \notin S$, there exists $A_z \in S$ such that $A_z > A_m$.

Since $A_m \notin S$, there had to be some point at which it was eliminated from further consideration to become a member of S . Let us say that A_m was eliminated when we were looking for A_{v+1} , having just chosen A_v .

If A_m was eliminated, then Step IV says that $W_v \leq W_m$ and since A_m was checked, $b_m > b_v$.

Thus, $A_v > A_m$ and there exists a chosen allocation which dominates A_m . //

It should be noted that the chosen sequence of allocations might have elements which would be dominated by an allocation in which the number of service channels exceeds $c + q$. But since we have said that $c + q$ is the maximum number of service channels which we will consider, there is no need to be concerned over this.

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